# Odd Prime Labeling of Snake Graphs

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**ABSTRACT**

Graph theory is one of the branches of mathematics which is concerned with the networks of points connected by lines. One of the most important research areas in graph theory is graph labeling, which dates back to the 1960s. Graph labeling is assigning integers to the vertices, edges, or both depending on conditions. Labeled graphs are helpful in mathematical models for a wide range of applications such as in coding theory, circuit theory, computer networks, and in cryptography as well. There are various types of graph labeling techniques in graph theory such as radio labeling, graceful labeling, prime labeling, antimagic labeling, and lucky labeling, etc. In this research, we use one of the variations of prime labeling called odd prime labeling of snake graphs. Recent works on odd prime labeling investigate about families of snake graphs, complete graphs, etc and there they discuss about one odd integer sequence only. In this research, we introduce odd prime labeling method for snake graphs for any odd integer sequence and we give a proof for it as well.

**KEYWORDS:** *snake graph, odd sequences, odd prime labeling, relatively prime*

## 1 INTRODUCTION

Graph theory is a branch of mathematics that studies the relationships between objects, called vertices or nodes, and the lines connecting them called edges. Graphs can be used to represent many different types of relationships, including ones between people, roads, and even computer networks. The concept of prime labeling was introduced by Roger Etringer in 1980 and then Meena.S and Kavitha.P worked more on it. The concept of odd prime labeling was introduced by Prajapati.U.M & Shah.K.P and then work on odd prime labeling by many, and researchers Meena.S , Kavitha.P and Gajalakshmi.G are some of them (Gajalakshmi & Meena, 2022), (Prajapati & Shah, n.d.). The odd prime labeling is one of the most recently researched versions and involves labeling the vertices with the set of odd integers $(1, 3, \dots, 2n-1)$ where $n$ is any positive integer $n \geq 1$, so that any two adjacent vertices are relatively prime.

Odd prime labeling work on simple, undirected, and connected graphs, which is an exciting research area in most recently with a considerable amount of literature on different types of graphs. A graph that allows odd prime labeling is called an odd prime graph.

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Let $u$, $v$ are adjacent vertices and then the greatest common divisor, $gcd\ (u\ ,v)$ must be 1 so that u and v are odd prime for all adjacent vertices $u$ and $v$. A snake graph is a type of mathematical graph in which the vertices are arranged in a "snake-like" pattern. This can refer to a variety of different configurations, but one common example is a graph in which the vertices are arranged in a series of consecutive rings, with the vertices on each ring connected to their neighbors in a snake-like pattern. In this type of snake graph, the vertices on the initial ring are connected to the vertices on the another ring. Snake graphs are sometimes used to model networks or other complex systems, and they can be studied using techniques from graph theory.(Carter & Fox, 2022)

Odd prime labeling is new labeling technique which was published by U. M. Prajapati & K. P. Shah around 2018s' and that is introduced in this paper. This paper focus on the Odd Prime Labeling of common graphs such as path graphs, complete bipartite graph, wheel, and other wheel-related graphs, including generalized Petersen graph $P_{n,2}$ (Prajapati & Shah, n.d.).

In 2022, the paper published by Gajalakshmi and Meena , On Odd Prime Labeling of Snake Related Graph prove that some snake-related graphs are odd prime graphs. The odd prime labeling of some new graphs was investigated in this paper. The results show that some snake-related graphs, such as the quadrilateral snake D($Q_n$), the triangular snake S($T_n$), the double triangular snake D($T_n$), the alternate triangular snake A( $T_n$), the triangular ladder T( $L_n$), and the open triangular ladder O(T$L_n$), are odd prime graphs (Gajalakshmi & Meena, 2022). The odd prime labeling problem has many practical applications, including in the design of secure communication networks and in the analysis of chemical compounds. It is also an active area of research in mathematics, with many open problems and ongoing developments. There are two main types of graphs: directed graphs and undirected graphs. In a directed graph, the edges have a direction and are called arcs. In an undirected graph, the edges do not have a direction and are simply called edges.

This paper focus on directed, simple, and connected snake graph $T_{n,m}$ which denotes m , length of the cycle & n, number of cycles. In this research work, investigate a generalized method for obtaining odd prime labeling for snake graph.

*Definition 1: Odd Prime Labeling*

A graph $G$ with vertex set $V(G)$ is said to have odd prime labeling if there exists an injective function $f\ :\ V(G)\ \rightarrow\ \{1,3,5,\dots,2|\,V(G)\,|-1\}$ such that for every edge $x,y\in E(G)$, $f(x)$ and $f(y)$ are relatively prime. A graph G that admits odd prime labeling is called an odd prime graph.

## 2 Materials & Methods

***Observation 1:***

An odd sequence is a sequence of numbers in which all the numbers are odd. An odd number is any integer that is not divisible by 2. For example, the sequence $\{\ 1,3,5,7,9,11\,..,(2n-1)\ \}$ is an odd sequence with the difference of any consecutive numbers is $2\ (d = 2)$, where d denotes the difference of the sequence for any $d = 2, 4, 6, .., 2k$ where k is any integer. In general, so to create an odd sequence, you would simply need to choose a series of numbers that fit this criterion. Odd sequences can be finite, with a fixed number of elements, or infinite, with an unlimited number of elements. Consecutive integers in any order of odd sequence are relatively prime.

Odd Sequence of $d = 2$ $\{1, 3, 5, 7, 9, 11 .., (2n - 1)\ \}$

Odd Sequence of $d = 4$ $\{1, 5\ , 9, 13, 17, \dots\ ,\ (4n - 3)\}$

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Odd Sequence of $d = 2k$ $\{1, (1 + 2k)\ , (1 + 4k), \dots\ (1 + 2nk - 2k)\}$

Likewise there are infinitely many odd sequences can be found.

***Observation 2:***

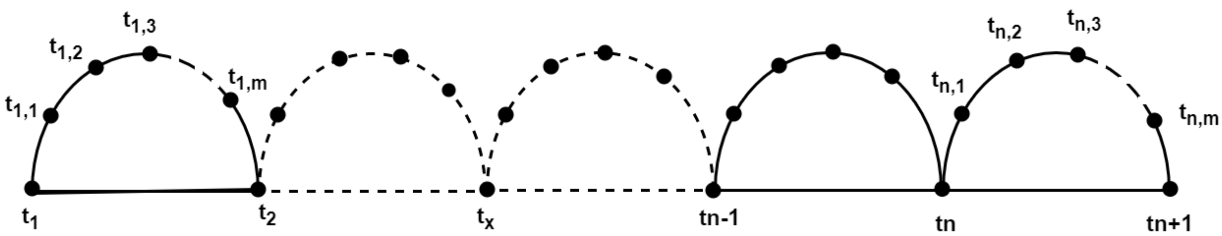
Properties of gcd:

For any non-zero integers a, b, c and any positive integer h, the following hold:

I. $gcd\ (a, a\ +\ 2h\ )\ =\ 1$ if $a$ is odd
II. $gcd\ (a, b)\ =\ gcd\ (a, a\ -\ b)$
III. $gcd\ (a, b)\ =\ gcd\ (a\ +\ cb\ , b)$

*Theorem:*

Snake graph $T_{n,m}$ is an odd prime graph where n denotes number of cycles in graph for any $n\ \geq\ 1$ and $n = 1, 2, 3, ..$ and where m denotes number of vertices in the cycle for any $m\ \geq\ 1$ and $m\ =\ 1, 2, 3, ..$ such that n and m are fixed for a particular given graph. n, m $\epsilon$ N (Natural number set).




*Proof.* We will label the cycles of graph sequentially from $t_1$ to $t_n$. Given a cycle $t_n$ consider the vertices in clockwise order as $t_{n,1}$, $t_{n,2}$, $t_{n,3}$, . . , $t_{n,m}$. Let $m$ be the number of vertices in the cycle and

$n$ be the number of cycles in graph. Create labeling starting with initial cycle $t_1$ in a snake graph by using any odd sequence found as mentioned above.
Label the sequence of vertices of cycle $t_n$,

$t_{n,1},\ t_{n,2},\ t_{n,3},\ t_{n,4},\ t_{n,5},\ \ldots, t_{n,m}$ respectively.

The vertices on cycle of the snake graph will be labeled with consecutive odd integers.

Similarly label the sequence of vertices of cycle $t_{n-1}$,

$t_{n-1,1},\ t_{n-1,2},\ t_{n-1,3},\ t_{n-1,4},\ t_{n-1,5},\ \ldots, t_{n-1,m}$ respectively.

The vertices on cycle of the snake graph will be labeled with consecutive odd integers.

The labels of the adjacent vertices of cycles in snake graph in both cases are relatively prime. Since Consecutive integers in any order of odd sequence is relatively prime, the labels of the adjacent vertices of cycles in snake graph are relatively prime.

Label the sequence of vertices of path graph $T_{n,m}$,

$t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ \ldots, t_n, t_{n+1}$ respectively.

The vertex pairs of snake graph $(t_1\ , t_2), (t_{2\ ,} t_3), .., (t_x\ , t_{x+1}), \ldots, (t_n\ , t_{n+1})$ are also adjacent to each other. As well as $(t_{x,}\ t_{x+1})$ vertex pair of snake graph adjacent to each other accordingly. Where $x$ is any cycle in snake graph $T_{n,m}$. Vertex pair of snake graph in above has to be relatively prime because they are adjacent with odd integers. Therefore, the pairs of labels that we have to be confirmed are relatively prime on the vertices $t_x$ and $t_{x+1}$ .

Label of vertex in path graph ( $t_1, t_2, .., t_x, t_{x+1}, .., t_n, t_{n+1}$ ) given by,

$$t_n = 1 + (m+1)(n-1)d \tag{1}$$

Label of vertex in cycle graph $t_{n,1},\ t_{n,2},\ t_{n,3},\ t_{n,4},\ t_{n,5},\ \ldots, t_{n,m}$ given by,

$$t_{n,m} = t_n + md \tag{2}$$

Where d denotes the difference of the sequence for any $d = 2, 4, 6, .., 2k$ where k is any integer.

Label of $t_x$ vertex, $$t_x = 1 + (m+1)(x-1)d \tag{3}$$

Label of $t_{x+1}$ vertex, $$t_{x+1} = 1 + (m+1)xd \tag{4}$$

$$gcd\ (t_x, t_{x+1}) = gdc\ (1 + (m+1)(x-1)d\ , 1 + (m+1)xd) \qquad \text{(Observation 2)}$$
$$= gdc\ (1 + (m+1)(x-1)d\ , (m+1)d)$$

$$= gdc\,(1 + (m+1)(x-1)d - (m+1)d(x-1)\,,(m+1)d\,)$$
$$= gdc\,(1\,,(m+1)d\,)$$
$$= 1.$$

Greatest Common Divisor ($gcd$) of vertex pair of snake graph ( $t_x, t_{x+1}$ ) is 1 which means vertex $t_x$ and $t_{x+1}$ are relatively prime. Therefore the vertex pairs of snake graph $(t_1\ ,t_2), (t_{2\ ,}t_3),\ldots,(t_x\ \ ,t_{x+1}),\ldots,(t_n\ \ ,t_{n+1})$ are also relatively prime.

Thus all snake graphs $T_{n,m}$ for any odd sequence are Odd Prime.

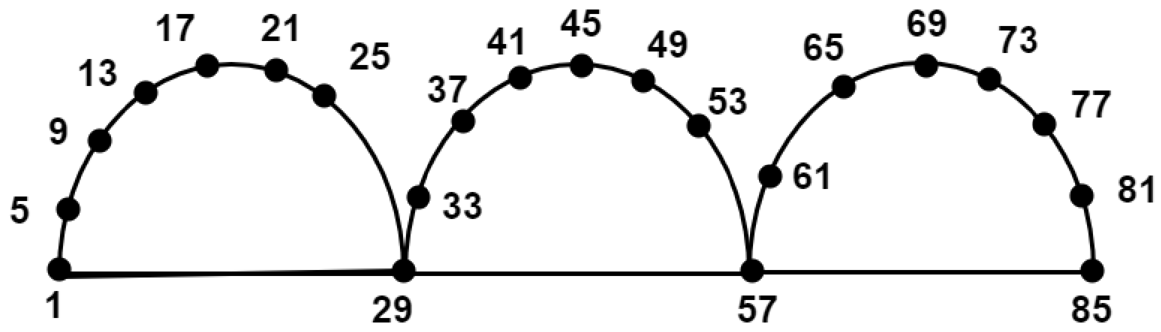



*Figure 2 : Odd Prime Labeling of graph $T_{3,6}$ using (1,5,9,13,..) sequence*

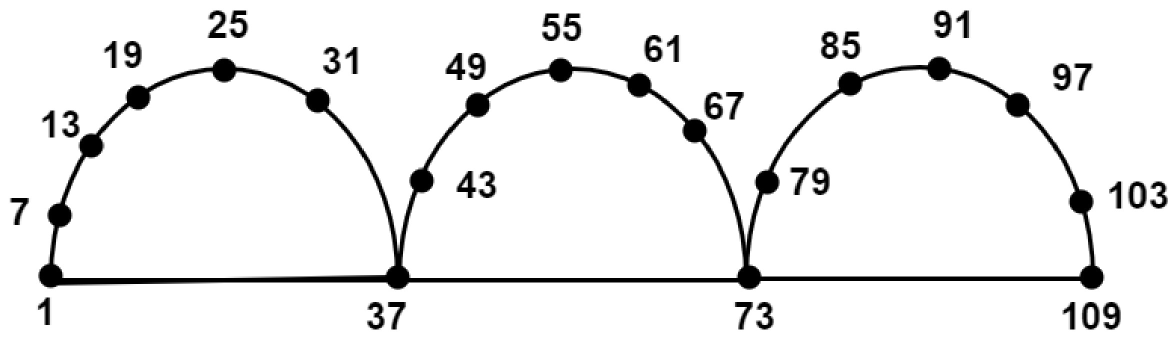



*Figure 3 : Odd Prime Labeling of graph $T_{3,5}$ using (1,7,13,..) sequence*

## 3 Results & Discussion

The odd prime labeling is one of the most recently researched versions and involves labeling the vertices with the set of odd integers so that any two adjacent vertices are relatively prime. A new labeling of snake graph has been introduced in this paper called an odd prime labeling of snake graph $T_{n,m}$ by using different types of odd sequences and introducing a general definition.

## 4 Conclusion

An odd sequence is a sequence of numbers in which all the numbers are odd. In general to create an odd sequence, simply need to choose a series of numbers that fit above criterion. The number of vertices in the cycle (m) and the number of cycles in the graph (n) are varying and the sequence that uses for labeling also can be varying. The snake graph can take different forms depending on the values of $n$ and $m$. According to the selection of an odd sequence, the labeling of the graph differs. In our paper, we prove that the snake graph can be labeled using odd prime labeling for any odd sequence and introduced a general theorem. Investigating similar theorems for other graphs is an open area of research.

## REFERENCES

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